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# ANALYTIC FORMULATION OF THREE WEDGE ANGLE CONSTRAINTS IN TRAJECTORY STUDIES



Mathematical Physics Branch
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CONTENTS

FIGURES

# ANALYTIC FORMULATION OF THREE WEDGE ANGLE CONSTRAINTS IN TRAJECTORY STUDIES

By Wayne O. Laszlo

## 1.0 SUMMARY AND INTRODUCTION

For many earth-moon and interplanetary trajectory studies, it may be desirable that the terminal parking orbit achieved by a vehicle be inclined at some prespecified wedge angle (also called angle of inclination) measured with respect to some reference coordinate system ($\vec{i},\vec{j},\vec{k}$ unit vectors). The occupied focus of the terminal parking orbit would be the center of the attracting planet. In the earth-moon trajectory optimization studies presently being undertaken by the Mathematical Physics Branch, Mission Planning and Analysis Division, a wedge angle constraint is being used in conjunction with a terminal circular lunar parking orbit.

The purpose of this document is to derive an analytic formulation of three wedge angle constraints for a vehicle's terminal parking orbit from a given state vector on that orbit. In the analysis presented, it is determined that the use of a particular wedge angle constraint for a vehicle's terminal parking orbit depends on the freedom desired in both the vehicle's direction of motion (posigrade or retrograde) on that orbit and in the orientation of the orbital plane. The three wedge angle constraints and their respective interpretations (including geometrical interpretations) are as follows.

First, when $\cos W - \cos \bar{W} = 0$ is satisfied, the vehicle's direction of motion corresponding to $\bar{W}$ will be fixed, and the orientation of the orbital plane will be left free to the extent that $\vec{H}$ rotates to form a one-nappe cone of angle of revolution $\bar{W}$ about the $\vec{k}$ axis. Second, when $\sin W - \sin \bar{W} = 0$ is satisfied, the vehicle's direction of motion corresponding to $\bar{W}$ will be left free, and the orientation of the orbital plane will be free to the extent that $\vec{H}$ rotates to form a two-nappe cone of angle of revolution $\bar{W}$ about the $\vec{k}$ axis. This constraint allows the maximum freedom in the vehicle's terminal parking orbit of the three constraints. Third, when $\bar{H} - \overline{(\vec{H})} = 0$ is satisfied, the vehicle's direction of motion corresponding to $\bar{W}$ will be fixed, and the orientation of the orbit plane will be fixed to the extent that there is no cone of revolution. This constraint allows the minimum freedom in the vehicle's terminal parking orbit of the three constraints.

## 2.0 SYMBOLS

| | |
|---|---|
| cos W | cosine of the wedge angle |
| cos $\bar{W}$ | cosine of the fixed wedge angle $\bar{W}$ |
| e | eccentricity of the terminal parking orbit |
| $\vec{H}$ | angular momentum vector associated with an orbit. The components in the $(\vec{i},\vec{j},\vec{k})$ coordinate system are $H_x$, $H_y$, $H_z$ |
| $\lvert\vec{H}\rvert$, H | magnitude of $\vec{H}$ (or $\sqrt{H_x^2 + H_y^2 + H_z^2}$) |
| $\overline{(\vec{H})}$ | fixed angular momentum vector that is to be achieved by satisfaction of the wedge angle constraint $\bar{H} - \overline{(\vec{H})} = 0$ |
| $(\vec{i},\vec{j},\vec{k})$ | mutually orthogonal unit vectors in the reference coordinate system |
| $(\vec{i}',\vec{j}',\vec{k}')$ | mutually orthogonal unit vectors in an alternate coordinate system |
| $\vec{M}$ | vector given by $\vec{M} = \vec{N} \times \vec{k}$. The components in the $(\vec{i},\vec{j},\vec{k})$ coordinate system are $M_x$, $M_y$, $M_z$ |
| $\lvert\vec{M}\rvert$, M | magnitude of M (or $\sqrt{M_x^2 + M_y^2 + M_z^2}$) |
| $\vec{N}$ | vector of ascending node given by $\vec{N} = \vec{k} \times \vec{H}$. The components in the $(\vec{i},\vec{j},\vec{k})$ coordinate system are $N_x$, $N_y$, $N_z$ |

| | |
|---|---|
| $\lvert\vec{N}\rvert$, N | magnitude of $\vec{N}$ (or $\sqrt{N_x^2 + N_y^2 + N_z^2}$) |
| $\vec{R}(x,y,z)$, $\vec{V}(u,v,w)$ | state vector of the vehicle on its terminal parking orbit. The position components are x,y,z, and the velocity components are u,v,w. Components are in the $(\vec{i},\vec{j},\vec{k})$ coordinate system. |
| sin W | sine of the wedge angle |
| sin $\bar{W}$ | sine of the fixed wedge angle $\bar{W}$ |
| V | angle determined by the dot product of $\vec{M}$ and $\vec{H}$, radians |
| W | wedge angle for any parking orbit determined by the dot product of $\vec{k}$ and $\vec{H}$, **radians** |
| $\bar{W}$ | fixed wedge angle that is to be achieved by satisfaction of the wedge angle constraint, radians |
| $\times$ | denotes cross product between two vectors |
| $\bar{\alpha}$ | fixed angle of inclination of the $(\vec{i}',\vec{j}')$ plane with respect to the $(\vec{i},\vec{j})$ plane, radians |
| $\bar{\Omega}$ | fixed longitude of ascending node (or angle between $\vec{i}$ and $\vec{i}'$), radians |
| $\cdot$ | denotes dot product between two vectors |

## 3.0 ANALYSIS

The wedge angle W for any parking orbit will be defined as the angle determined by the dot $(\cdot)$ product of $\vec{k}$ and $\vec{H}$, where $\vec{H}$ is the angular momentum vector associated with the orbit. The wedge angle is also given by the solid angle between the reference $(\vec{i},\vec{j})$ plane and the parking orbit plane. By the geometry of figure 1, it can be concluded that $0 \leq W \leq \pi$. Furthermore, for $0 \leq W < \frac{\pi}{2}$, $H_z > 0$ and the orbit will be posigrade; while for $\frac{\pi}{2} < W \leq \pi$, $H_z = 0$ and the orbit will be retrograde. If $W = \frac{\pi}{2}$, then $H_z = 0$ and the orbit may be called a polar orbit.

To derive an analytic formulation of one wedge angle constraint for a vehicle's terminal parking orbit, it is assumed that $\vec{R}(x,y,z)$, $\vec{V}(u,v,w)$ represents a state vector of the vehicle for that orbit. Let $\bar{W}$ (a fixed angle in radians) be the wedge angle that is to be achieved. The vector $\vec{H} = \vec{R} \times \vec{V}$ ($\times$ indicates cross product) is computed. The determinant given by equation (1) defines $\vec{H}$.

$$H = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ u & v & w \end{pmatrix} = H_x\vec{i} + H_y\vec{j} + H_z\vec{k} \tag{1}$$

Thus, the components of $H$ can be written according to equations (2), (3), and (4).

$$H_x = yw - zv \tag{2}$$

$$H_y = zu - xw \tag{3}$$

$$H_z = xv - yu \tag{4}$$

If $\vec{H} = 0$ (i.e., $H_x = H_y = H_z = 0$), then $\vec{R}$ is parellel to $\vec{V}$, which implies rectilinear motion and impact at the center of the planet. Although such an event is very unlikely, the wedge angle constraint will diverge if $\vec{H} = \vec{0}$, and a numerical value of the constraint would have to be specified.

If $\vec{H} \neq \vec{0}$, cos W can be computed from equation (5).

$$\frac{\vec{k} \cdot \vec{H}}{|\vec{k}|\ |\vec{H}|} = \cos W \tag{5}$$

From the latter equation, note that $W$ is the smallest angle between $\vec{k}$ and $\vec{H}$ and that

$$\cos W = \frac{H_z}{H} \tag{6}$$

where $H = |\vec{H}|$. Then, given $\bar{W}$ and therefore $\cos \bar{W}$, the wedge angle constraint may be written as

$$\cos W - \cos \bar{W} = 0 \tag{7}$$

In terms of $(\vec{R},\vec{V})$, equation (7) may be written as follows.

$$\frac{xv - yu}{\sqrt{H_x^2 + H_y^2 + H_z^2}} - \cos \bar{W} = 0 \qquad (8)$$

where $H_x$, $H_y$, $H_z$ are given in terms of x, y, z, u, v, w by equations (2), (3), and (4). Note that because $0 \leq W \leq \pi$, this constraint [given by equations (7) or (8)] is precisely equivalent to the constraint $W - \bar{W} = 0$. To give a geometrical interpretation of the wedge angle constraint [eq. (7)], recall that the wedge angle W for an orbit carries with it the vehicle's direction of motion (posigrade or retrograde) in the sense that $0 \leq W < \frac{\pi}{2}$ implies posigrade motion while $\frac{\pi}{2} < W \leq \pi$ implies retrograde motion. Then, because equation (7) is equivalent to the constraint $W - \bar{W} = 0$ ($\bar{W}$ an input constant), it follows that when $\cos W - \cos \bar{W} = 0$ the vehicle will have achieved a terminal parking orbit with the fixed wedge angle $\bar{W}$ as well as with the fixed direction of motion corresponding to $\bar{W}$. However, while equation (7) does constrain the angle between $\vec{k}$ and $\vec{H}$, namely W, the direction of $\vec{H}$ has some freedom of rotation. In other words, the analytic formulation given by equation (7) yields some freedom of orientation of the terminal parking orbit plane. In fact, the geometrical interpretation of equation (7) (fig. 3) is that $\vec{H}$ rotates to form a one-nappe cone of angle of revolution $\bar{W}$ about the $\vec{k}$ axis. The freedom of orientation of the orbital plane is also shown in figure 3 in that orbits 1 and 2 each have permissible planar orientations which satisfy equation (7). In trajectory optimization studies in which equation (7) is used, the optimal (i.e., minimum fuel) orientation of the terminal parking orbit plane will be attained.

Suppose it is desired to derive an analytic formulation of a wedge angle constraint which when satisfied will leave free the vehicle's direction of motion corresponding to $\bar{W}$ (posigrade or retrograde). Such a constraint would result in the same freedom of orientation of the orbital plane as equation (7). The constraint that will permit satisfaction of the latter two functions is given by equation (9).

$$\sin W - \sin \bar{W} = 0 \qquad (9)$$

For $0 \leq W \leq \pi$, it follows that if $\sin W = \sin \bar{W}$ ($\bar{W}$ an input constant), then $W = \bar{W}$ or $W = \pi - \bar{W}$. Therefore, if $0 \leq \bar{W} < \frac{\pi}{2}$ (posigrade motion), then conceivably $W = \pi - \bar{W}$ (retrograde motion). It may then be said that when equation (9) is satisfied, the vehicle's direction of motion corresponding to $\bar{W}$ is free, and there is the same freedom of orientation of the orbital plane as in the previous formulation.

To derive the analytic formulation of the wedge angle constraint [eq. (9)], from $\vec{R}(x,y,z)$, $\vec{V}(u,v,w)$, $\vec{H}$ is computed by equation (1) with components given by equations (2), (3), and (4). Then, if $\vec{H} \neq \vec{0}$, $\vec{N} = \vec{k} \times \vec{H}$, the vector of ascending node is computed. The vector $\vec{N}$ is given by the determinant

$$\vec{N} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ H_x & H_y & H_z \end{pmatrix} = N_x\vec{i} + N_y\vec{j} + N_z\vec{k} \tag{10}$$

Thus, the components of $\vec{N}$ can be written according to equations (11), (12), and (13).

$$N_x = -H_y \tag{11}$$

$$N_y = H_x \tag{12}$$

$$N_z = 0 \tag{13}$$

If $\vec{N} = \vec{0}$ (i.e., $-H_y = H_x = 0$, or $\vec{k}$ is parallel to $\vec{H}$), then an orbit is obtained with either a 0 wedge angle (inclination) if $H_z > 0$ or with a $\pi$ wedge angle (inclination) if $H_z < 0$. However, if $\vec{N} \neq \vec{0}$ (i.e., either $H_x \neq 0$ or $H_y \neq 0$ or both), then $\vec{M} = \vec{N} \times \vec{k}$ is computed. The vector $\vec{M}$ is given by the determinant

$$\vec{M} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ N_x & N_y & 0 \\ 0 & 0 & 1 \end{pmatrix} = M_x\vec{i} + M_y\vec{j} + M_z\vec{k} \tag{14}$$

Thus, the components of $\vec{M}$ can be written according to equations (15), (16), and (17).

$$M_x = N_y = H_x \tag{15}$$

$$M_y = -N_x = H_y \tag{16}$$

$$M_z = 0 \tag{17}$$

Next, it can be shown that $\vec{M}$ is a vector in both the $(\vec{i},\vec{j})$ plane and in the plane determined by the noncollinear vectors $\vec{k}$ and $\vec{H}$. To verify this result, recall that $\vec{M}$ must be perpendicular to $\vec{N}$ and $\vec{k}$ from vector analysis because $\vec{M} = \vec{N} \times \vec{k}$. Further, because $\vec{N} = \vec{k} \times \vec{H}$, it can be concluded that all vectors in the plane determined by vectors $\vec{k}$ and $\vec{H}$ are perpendicular to $\vec{N}$, and any vector perpendicular to $\vec{N}$ must lie in this plane. Therefore, $\vec{M}$ lies in the plane determined by vectors $\vec{k}$ and $\vec{H}$. With the latter results in mind, the angle $V$ will be the angle determined by the dot $(\cdot)$ product of $\vec{M}$ and $\vec{H}$, that is, the smallest angle between $\vec{M}$ and $\vec{H}$. As seen in figure 2, if $0 < W \leq \frac{\pi}{2}$, then

$$W + V = \frac{\pi}{2} \tag{18}$$

Further, if $\frac{\pi}{2} < W < \pi$, then

$$V = W - \frac{\pi}{2} \tag{19}$$

For either case [eq. (18) or eq. (19)], the cosine of $V$ can be written as follows.

$$\cos V = \frac{\vec{M} \cdot \vec{H}}{|\vec{M}|\ |\vec{H}|} = \sin W$$

where

$$|\vec{M}| = M = \sqrt{M_x^2 + M_y^2 + M_z^2}$$

and

$$|\vec{H}| = H = \sqrt{H_x^2 + H_y^2 + H_z^2} \tag{20}$$

The rewriting of equation (20) results in equation (21)

$$\sin W = \frac{M_x H_x + M_y H_y + M_z H_z}{MH} \tag{21}$$

and the use of equations (15), (16), and (17) results in equation (22).

$$\sin W = \frac{H_x^2 + H_y^2}{\sqrt{H_x^2 + H_y^2}\,\sqrt{H_x^2 + H_y^2 + H_z^2}} \tag{22}$$

Therefore, in terms of $(\vec{R},\vec{V})$, equation (9) may be written as

$$\frac{H_x^2 + H_y^2}{\sqrt{H_x^2 + H_y^2}\,\sqrt{H_x^2 + H_y^2\ H_z^2}} - \sin \bar{W} = 0 \tag{23}$$

where $H_x$, $H_y$, $H_z$ are given in terms of x, y, z, u, v, w by equations (2), (3), and (4).

The geometrical interpretation of equation (9) (fig. 4) is that $\vec{H}$ rotates to form a two-nappe cone of angle of revolution $\bar{W}$ about the $\vec{k}$ axis. The freedom of orientation of the orbital plane is shown in figure 4 in that orbits 1 and 2 each have permissible planar orientations which satisfy equation (9).

Finally, suppose it is desired to derive an analytic formulation of a wedge angle constraint which when satisfied will fix the orientation of the orbital plane and will fix the vehicle's direction of motion corresponding to $\bar{W}$. The constraint that will result in satisfaction of the latter two functions is given by

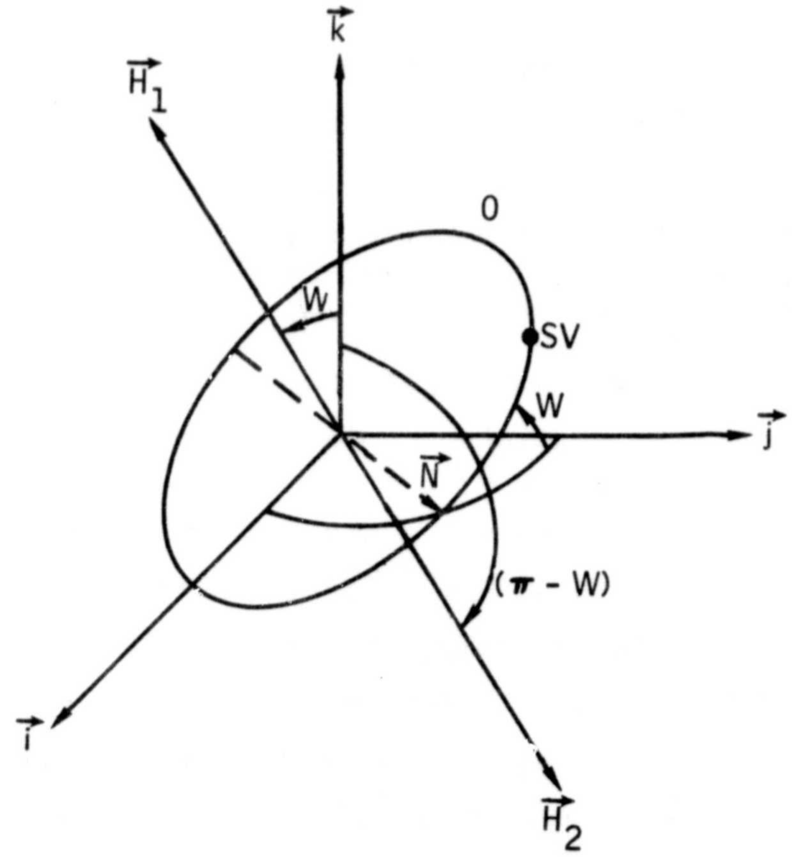
$$\vec{H} - \overline{(\vec{H})} = 0 \tag{24}$$

where $\vec{H}$ is given in terms of x, y, z, u, v, w by equations (2), (3), and (4).

The geometrical interpretation of equation (24) is given in figure 5. As seen by the figure, there is no cone of revolution to correspond to equation (24), and orbit 1 has the only permissible planar orientation that satisfies equation (24).

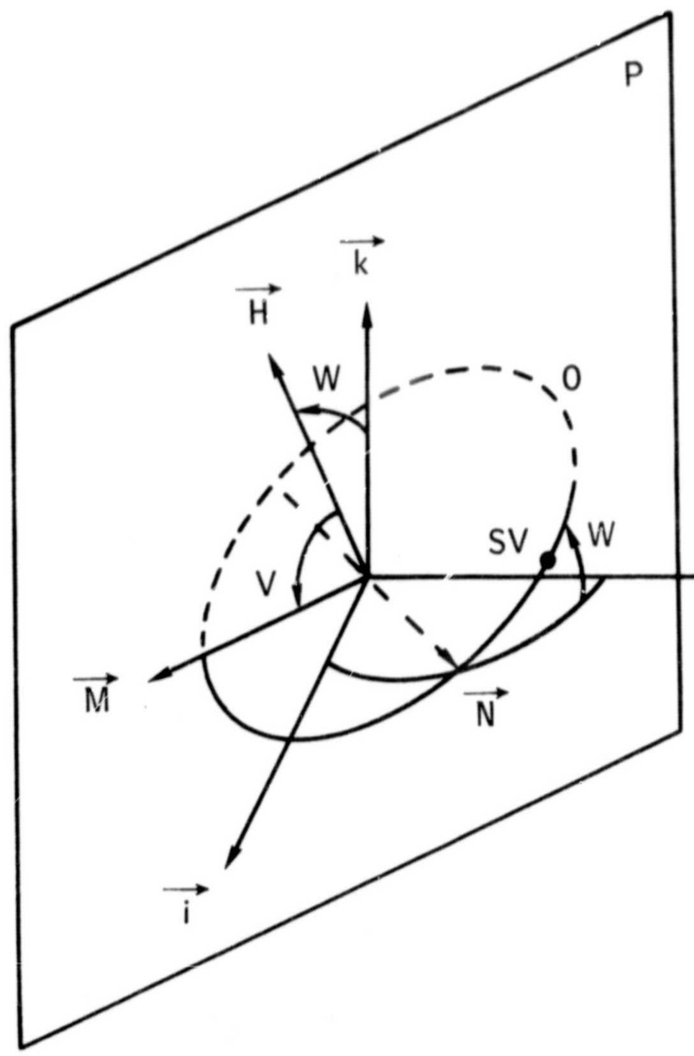
# 4.0 CONCLUDING REMARKS

Three comments are in order. First, in the event that $\bar{W} = 0, \frac{\pi}{2}$, or $\pi$, two of the three constraints are equivalent. For $\bar{W} = 0$, $\vec{H} - \overline{(\vec{H})} = 0$, and $\cos W - \cos \bar{W} = 0$ are equivalent; for $\bar{W} = \frac{\pi}{2}$, $\cos W - \cos \bar{W} = 0$ and $\sin W - \sin \bar{W} = 0$ are equivalent; and for $\bar{W} = \pi$, $\vec{H} - \overline{(\vec{H})} = 0$ and $\cos W - \cos \bar{W} = 0$ are again equivalent. Second, to insure that the vehicle's terminal parking orbit is either elliptical or circular, a constraint of the form $0 \leq e < 1$ (e denotes eccentricity of the terminal orbit) may be used in conjunction with the desired wedge angle constraint. Third, note that the wedge angle $W$ for any parking orbit is measured with respect to some reference ($\vec{i},\vec{j},\vec{k}$ unit vectors) coordinate system. However, suppose it is desired that the vehicle achieve a parking orbit with wedge angle $\bar{W}$ measured with respect to some alternate coordinate system ($\vec{i}',\vec{j}',\vec{k}'$ unit vectors). If the vehicle is assumed to be located at some state vector $\vec{R}(x,y,z)$, $\vec{V}(u,v,w)$ in space, it will be necessary to rotate the $(\vec{i},\vec{j},\vec{k})$ system into the $(\vec{i}',\vec{j}',\vec{k}')$ system and to express the state vector $\vec{R},\vec{V}$ in terms of the alternate $(\vec{i}',\vec{j}',\vec{k}')$ system. The latter process could consist of two rotations, the first through the angle $\bar{\Omega}$ from $\vec{i}$ to $\vec{i}'$ ($\Omega$ called the longitude of ascending node) and the second through the angle $\bar{\alpha}$ from the $(\vec{i},\vec{j})$ reference plane to the $(\vec{i}',\vec{j}')$ reference plane [$\bar{\alpha}$ called the angle of inclination of the $(\vec{i}',\vec{j}')$ plane with respect to the $(\vec{i},\vec{j})$ plane].



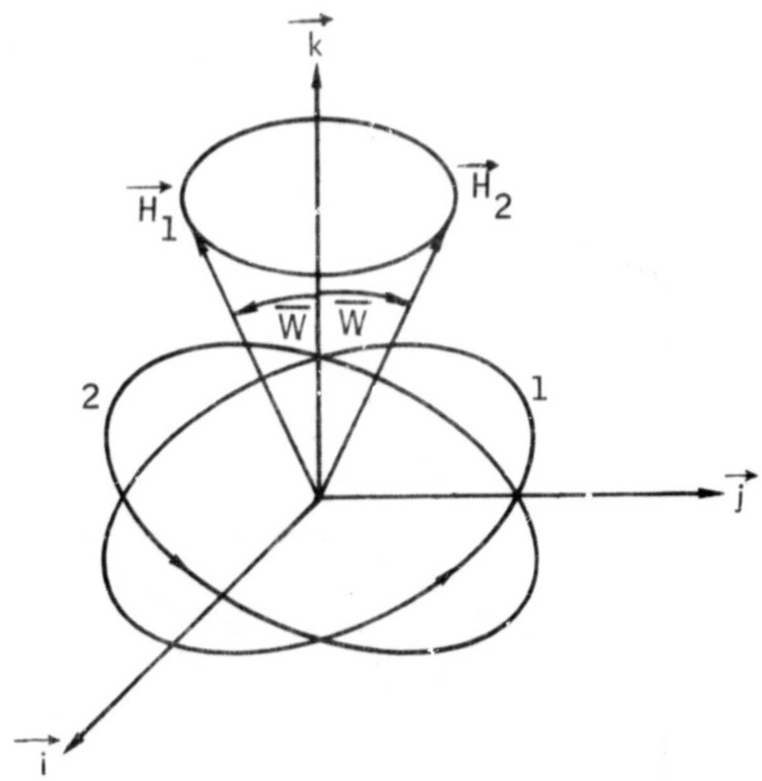

| | |
|---|---|
| 0 | VEHICLE'S TERMINAL PARKING ORBIT |
| SV | TYPICAL STATE VECTOR ON ORBIT 0 |
| $\vec{N}$ | THE VECTOR OF ASCENDING NODE |
| $\vec{H}_1$ | THE ANGULAR MOMENTUM VECTOR ASSOCIATED WITH POSIGRADE MOTION ON ORBIT 0 |
| W | THE WEDGE ANGLE ASSOCIATED WITH POSIGRADE MOTION ON ORBIT 0 |
| $\vec{H}_2$ | THE ANGULAR MOMENTUM VECTOR ASSOCIATED WITH RETROGRADE MOTION ON ORBIT 0 |
| $(\pi - W)$ | THE WEDGE ANGLE ASSOCIATED WITH RETROGRADE MOTION ON ORBIT 0 |

Figure 1.- Geometrical interpretation of the wedge angle in reference coordinate system.




| | |
|---|---|
| O | VEHICLE'S TERMINAL PARKING ORBIT |
| SV | TYPICAL STATE VECTOR ON ORBIT O |
| $\vec{N}$ | THE VECTOR OF ASCENDING NODE GIVEN BY $\vec{N} = \vec{k} \times \vec{H}$ |
| $\vec{M}$ | THE VECTOR GIVEN BY $\vec{M} = \vec{N} \times \vec{k}$ |
| $\vec{H}$ | THE ANGULAR MOMENTUM VECTOR ASSOCIATED WITH POSIGRADE MOTION ON ORBIT O |
| W | THE WEDGE ANGLE ASSOCIATED WITH POSIGRADE MOTION ON ORBIT O |
| V | THE ANGLE DETERMINED BY THE DOT $(\cdot)$ PRODUCT OF $\vec{M}$ AND $\vec{H}$ |
| P | PLANE CONTAINING VECTORS $\vec{k}$, $\vec{H}$, AND $\vec{M}$ |

Figure 2.- Vector geometry.




1 ORBIT WITH ASSOCIATED ANGULAR MOMENTUM VECTOR $\vec{H}_1$

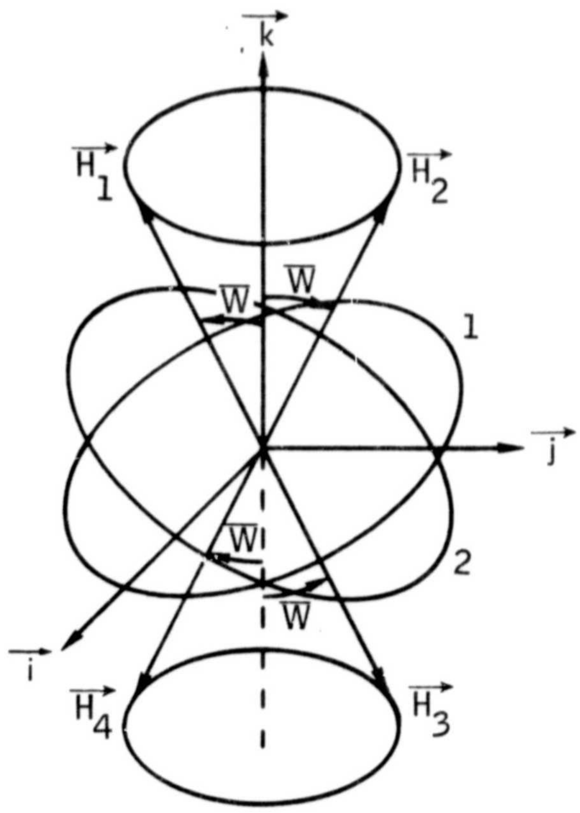
2 ORBIT WITH ASSOCIATED ANGULAR MOMENTUM VECTOR $\vec{H}_2$

$\vec{H}_1$ THE ANGULAR MOMENTUM VECTOR ASSOCIATED WITH POSIGRADE MOTION ON ORBIT 1

$\vec{H}_2$ THE ANGULAR MOMENTUM VECTOR ASSOCIATED WITH POSIGRADE MOTION ON ORBIT 2

$\overline{W}$ THE FIXED WEDGE ANGLE THAT IS TO BE ACHIEVED BY SATISFYING THE WEDGE ANGLE CONSTRAINT $\cos W - \cos \overline{W} = 0$

Figure 3.- Geometrical interpretation of the wedge angle constraint $\cos W - \cos \overline{W} = 0$.




1 ORBIT WITH ASSOCIATED ANGULAR MOMENTUM VECTOR EITHER $\vec{H}_1$ OR $\vec{H}_3$

2 ORBIT WITH ASSOCIATED ANGULAR MOMENTUM VECTOR EITHER $\vec{H}_2$ OR $\vec{H}_4$

$\vec{H}_1$ THE ANGULAR MOMENTUM VECTOR ASSOCIATED WITH POSIGRADE MOTION ON ORBIT 1

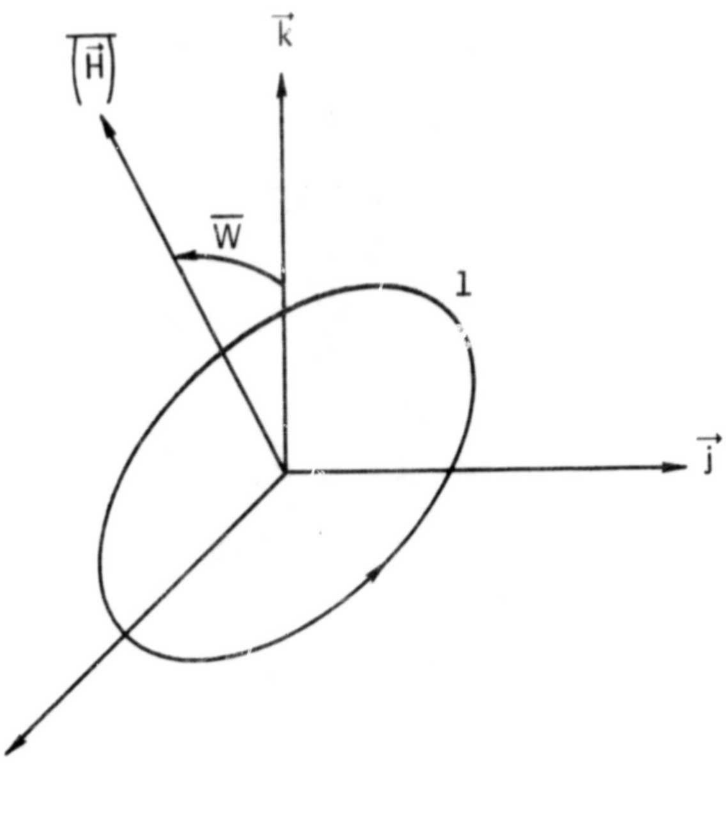
$\vec{H}_3$ THE ANGULAR MOMENTUM VECTOR ASSOCIATED WITH RETROGRADE MOTION ON ORBIT 1

$\vec{H}_2$ THE ANGULAR MOMENTUM VECTOR ASSOCIATED WITH POSIGRADE MOTION ON ORBIT 2

$\vec{H}_4$ THE ANGULAR MOMENTUM VECTOR ASSOCIATED WITH RETROGRADE MOTION ON ORBIT 2

$\overline{W}$ THE FIXED WEDGE ANGLE THAT IS TO BE ACHIEVED BY SATISFYING THE WEDGE ANGLE CONSTRAINT $\sin W - \sin \overline{W} = 0$

Figure 4.- Geometrical interpretation of the wedge angle constraint $\sin W - \sin \overline{W} = 0$.




| | |
|---|---|
| 1 | ORBIT WITH ASSOCIATED ANGULAR MOMENTUM VECTOR $\overline{\vec{H}}$ |
| $\overline{\vec{H}}$ | THE FIXED ANGULAR MOMENTUM VECTOR THAT IS TO BE ACHIEVED BY SATISFYING THE WEDGE ANGLE CONSTRAINT $\vec{H} - \overline{\vec{H}} = 0$ |
| $\overline{W}$ | THE FIXED WEDGE ANGLE THAT IS TO BE ACHIEVED BY SATISFYING THE WEDGE ANGLE CONSTRAINT $\vec{H} - \overline{\vec{H}} = 0$ |

Figure 5.- Geometrical interpretation of the wedge angle constraint $\vec{H} - \overline{\vec{H}} = 0$.

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1. Williamson, John Bruce: Computation of Euler Angles to Transform Between Spacecraft Local Coordinate Systems. MSC memo 66-FM46-227, August 4, 1966.